EE 505

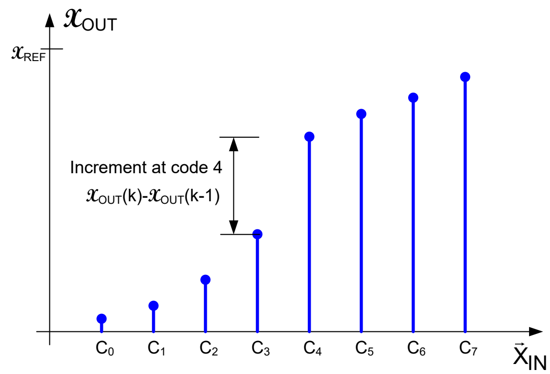
Lecture 4

Quantization Noise
Spectral Characterization

CORRECTION from Last Lecture

Differential Nonlinearity (DAC)

Nonideal DAC



Theorem: The INL_k of a DAC (when corrected for gain error and offset) can be obtained from the DNL by the expression

$$INL_k = \sum_{i=1}^{k} DNL(i)$$

Caution: Be careful about using this theorem to measure the INL since errors in DNL measurement (or simulation) can accumulate

Corollary: The DNL of a DAC (when corrected for gain error and offset) can be expressed as

$$DNL(k)=INL_k-INL_{k-1}$$

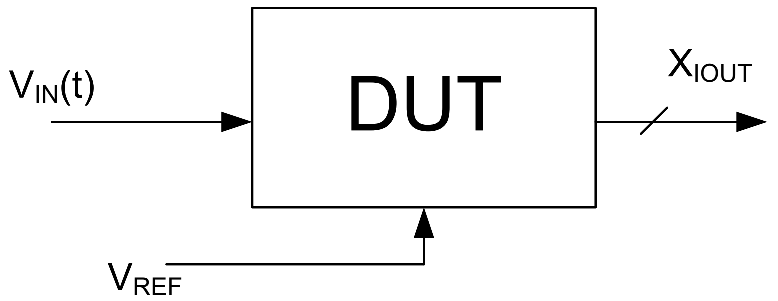
Commercial Data Converter Update (Jan 25, 2023)

Analog Devices					
	Flash				7
	MS				2
	Pipe				566
	SAR				863
	Delta Sigma				255
				Total	1693
Texas Instruments					
	Flash				3
	Two-Step				6
	Folding				64
	Pipe				296
	SAR				428
	Delta Sigma				195
	Not Specified				2
				Total	994

Note: Based upon reported part numbers. ADI lists some parts in multiple performance categories so some are listed more than once. Both have variants of one component listed with unique part numbers

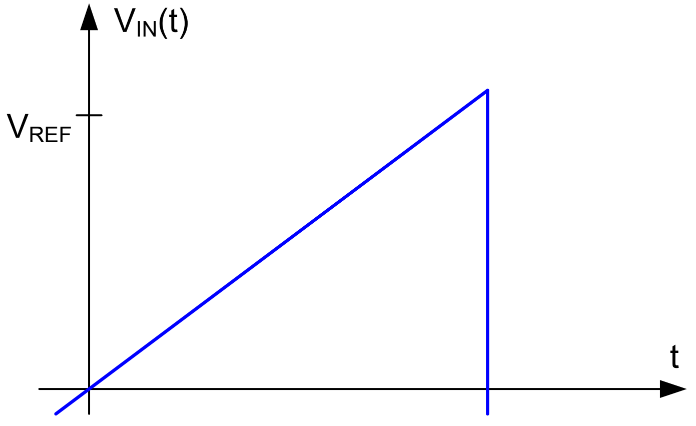
Linearity Measurements (testing)

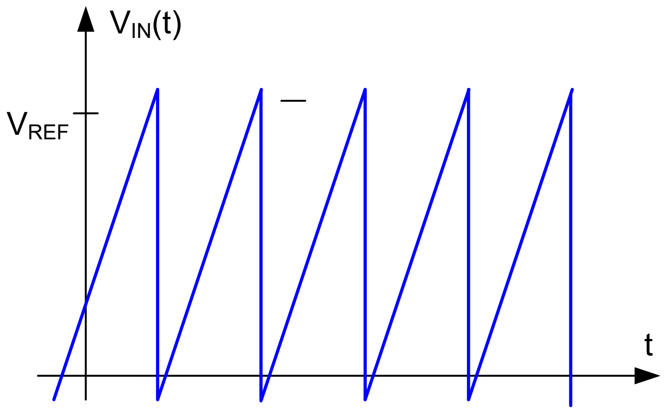
Consider ADC



Linearity testing often based upon code density testing

Code density testing:

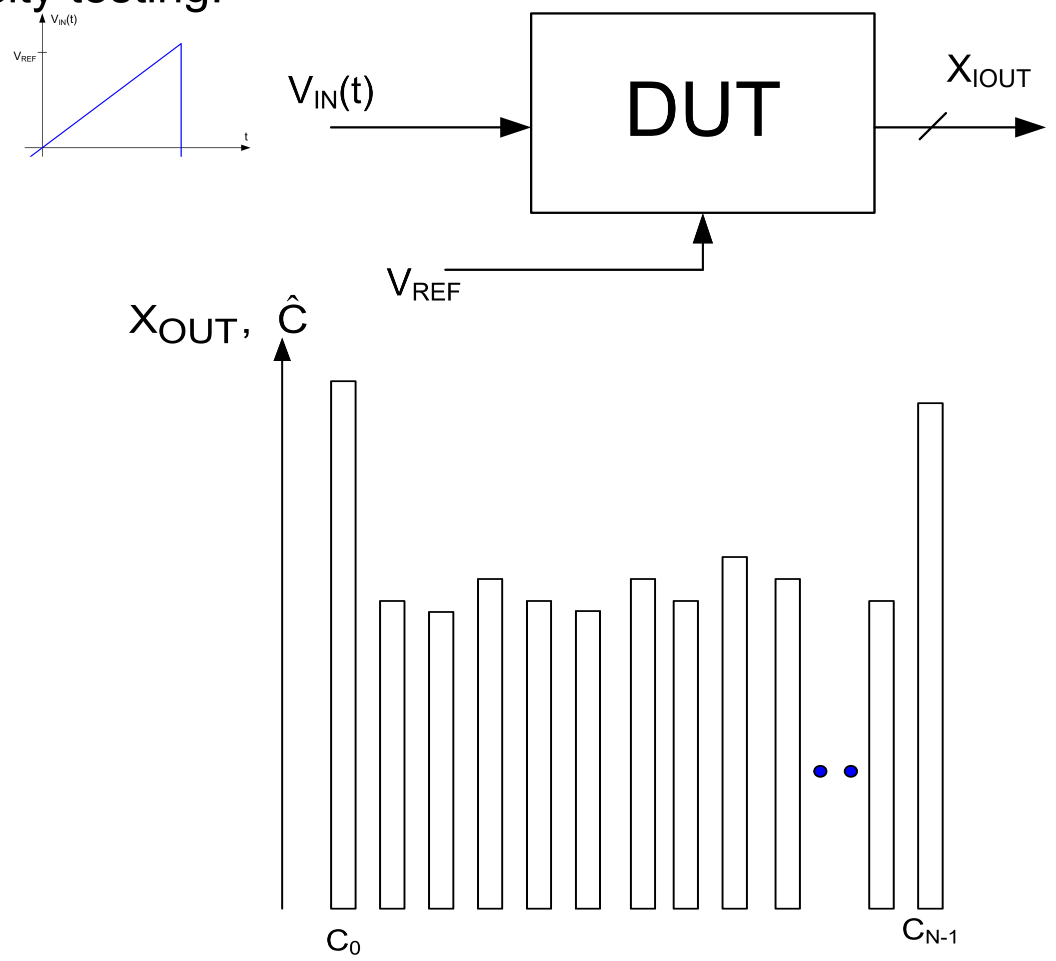




Ramp or multiple ramps often used for excitation
Linearity of test signal is critical (typically 3 or 4 bits more linear than DUT)

Linearity Measurements (testing)

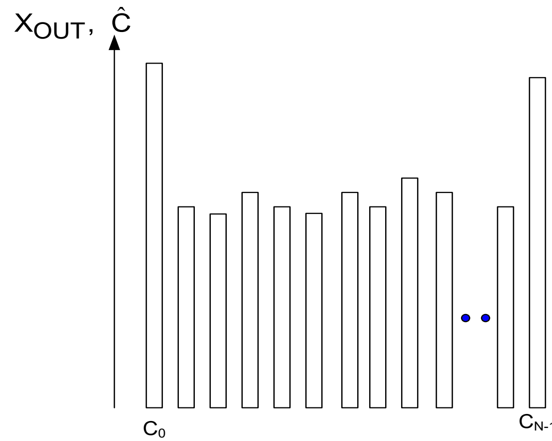
Code density testing:



- First and last bins generally have many extra counts (and thus no useful information)
- Typically average 16 or 32 hits per code

Linearity Measurements (testing)

Code density testing:



$$\bar{C} = \frac{\sum_{i=1}^{N-2} \hat{C}_i}{N-2}$$

$$DNL_i = \frac{\hat{C}_i - \bar{C}}{\bar{C}}$$

$$INL_i = \begin{cases} 0 & i=0,N-2 \\ \dfrac{\left[\sum_{k=1}^{i} \hat{C}_k\right] - i\bar{C}}{\bar{C}} & 1 \le i \le N-3 \end{cases}$$

$$DNL = \max_{1 \le i \le N-2} \left\{ \left| DNL_i \right| \right\}$$

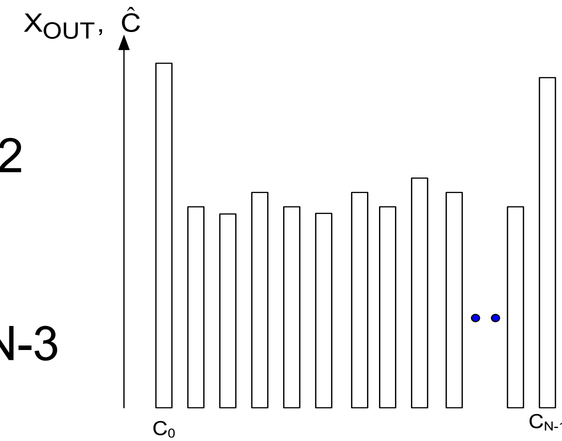
$$INL = \max_{1 \le i \le N-3} \left\{ \left| INL_i \right| \right\}$$

Linearity Measurements (testing)

Code density testing:

$$DNL_i = \frac{\hat{C}_i - \bar{C}}{\bar{C}} \qquad INL_i = \begin{cases} 0 & i=0, N-2 \\ \dfrac{\left[\sum\limits_{k=1}^{i} \hat{C}_k\right] - i\bar{C}}{\bar{C}} & 1 \le i \le N-3 \end{cases}$$

$$DNL = \max_{1 \le i \le N-2} \left\{ \left| DNL_i \right| \right\} \qquad INL = \max_{1 \le i \le N-3} \left\{ \left| INL_i \right| \right\}$$



- **Code Density Measurements are Indirect Measurements of the INL and DNL**
- **Can give very wrong information under some nonmonotone missing code scenarios**
- **Often use an average of 16 or 32 samples per code**
- **Measurement noise often 1 lsb or larger but averages out**
- **Sometimes use good sinusoidal waveform but must correct code density for this distinction**
- **Full code-density testing is costly for high-resolution low-speed data converters because of data acquisition costs**
- **Reduced code testing using servo methods is often a less costly alternative but may miss some errors**

Performance Characterization of Data Converters

- Static characteristics
  - Resolution
  - Least Significant Bit (LSB)
  - Offset and Gain Errors
  - Absolute Accuracy
  - Relative Accuracy
  - Integral Nonlinearity (INL)
  - Differential Nonlinearity (DNL)
  - Monotonicity (DAC)
  - Missing Codes (ADC)
  - Quantization Noise
  - Low-f Spurious Free Dynamic Range (SFDR)
  - Low-f Total Harmonic Distortion (THD)
  - Effective Number of Bits (ENOB)
  - Power Dissipation

Quantization Noise

- DACs and ADCs generally quantize both amplitude and time
- If converting a continuous-time signal (ADC) or generating a desired continuous-time signal (DAC) these quantizations cause a difference in time and amplitude from the desired signal
- First a few comments about Noise

What is Noise in a data converter?

Noise is a term applied to some nonideal effects of a data converter

Precise definition of noise is probably not useful

Some differences in views about what nonideal characteristics of a data converter should be referred to as noise

Types of noise:

- Random perturbations in V or I due to movement of electrons in electronic circuits
- Interfering signals generated by other systems
- Interfering signals generated by a circuit or system itself
- Error signals associated with imperfect signal processing algorithms or circuits
  - Quantization noise
  - Sample Jitter
  - Harmonic Distortion

Noise

All of these types of noise are present in data converters and are of concern when designing most data converters

Can not eliminate any of these noise types but with careful design can manage their effects to certain levels

Noise (in particular the random noise) is often the major factor limiting the ultimate performance potential of many if not most data converters

Noise

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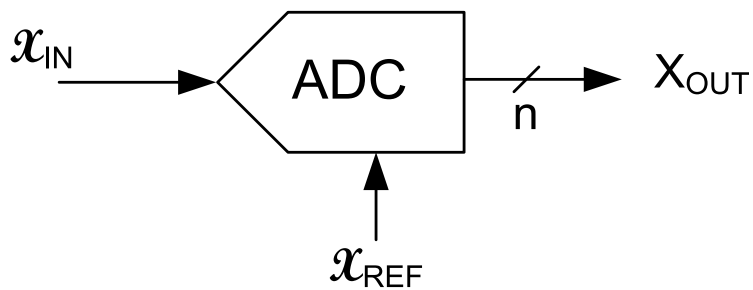
Quantization noise is a significant component of this noise in ADCs and DACs and is present even if the ADC or DAC is ideal

Only the first type is associated with random variations but from a performance limitation viewpoint, all

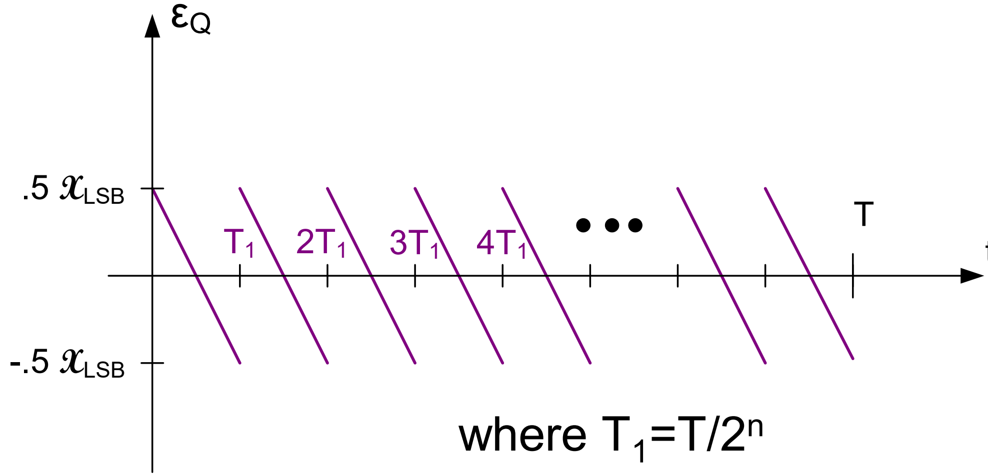
Quantization Noise in ADC

(same concepts apply to DACs)

Consider an Ideal ADC with first transition point at $0.5X_{LSB}$



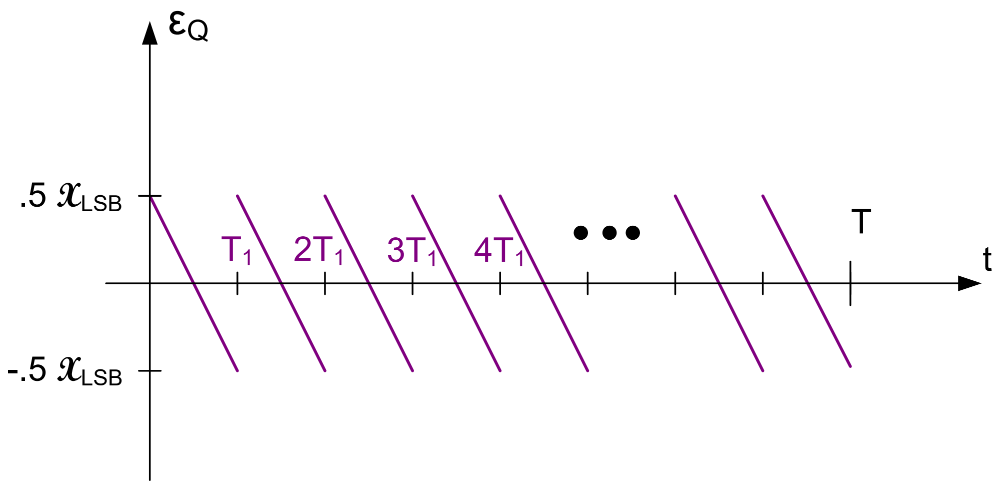
If the input is a low frequency sawtooth waveform of period T that goes from 0 to X_{REF} , the error signal in the time domain will be:



where $T_1=T/2^n$

This time-domain waveform (after dc offset is removed) is termed the Quantization Noise for the ADC with a sawtooth (or triangular) input

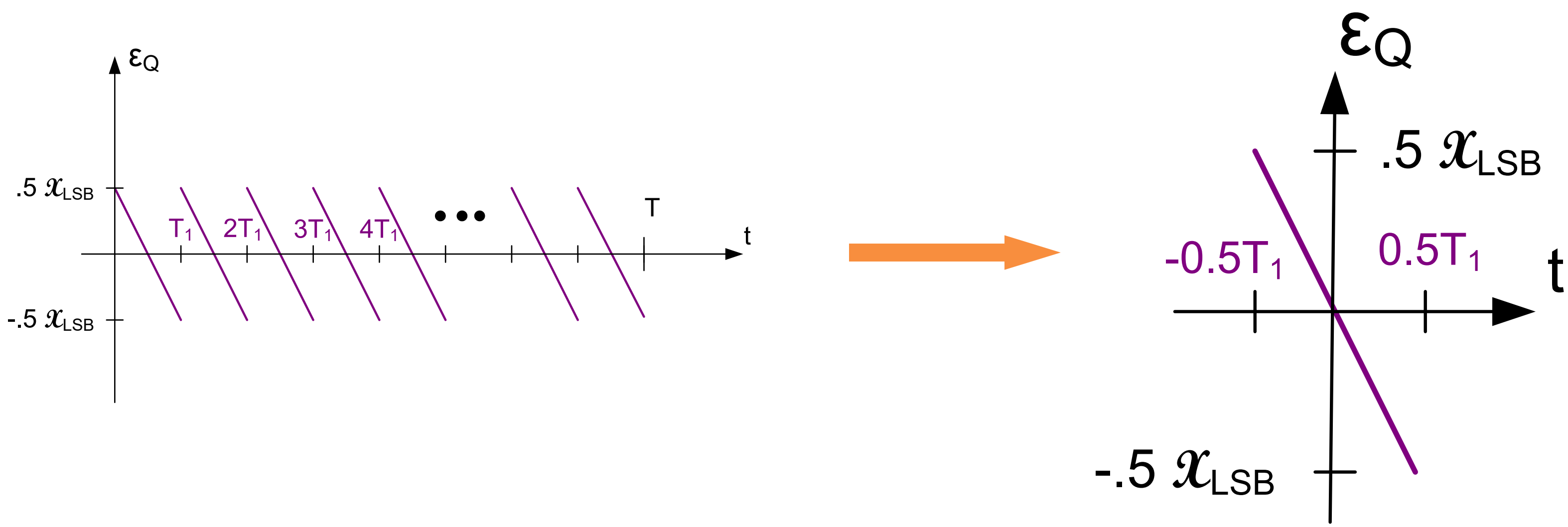
Quantization Noise in ADC



For large n, this periodic waveform "behaves" much like a random noise source that is uncorrelated with the input and can be characterized by its RMS value which can be obtained by integrating over any interval of length T_1. For notational convenience, shift the waveform to the left by $T_1/2$ units

$$E_{RMS} = \sqrt{\frac{1}{T_1}\int_{-T_1/2}^{T_1/2} \varepsilon_Q^{2}(t)\,dt}$$

Quantization Noise in ADC



$$E_{RMS} = \sqrt{\frac{1}{T_1}\int_{-T_1/2}^{T_1/2} \varepsilon_Q^{\,2}(t)\,dt}$$

In this interval, ε_Q can be expressed as

$$\varepsilon_Q(t) = -\left(\frac{X_{LSB}}{T_1}\right)t$$

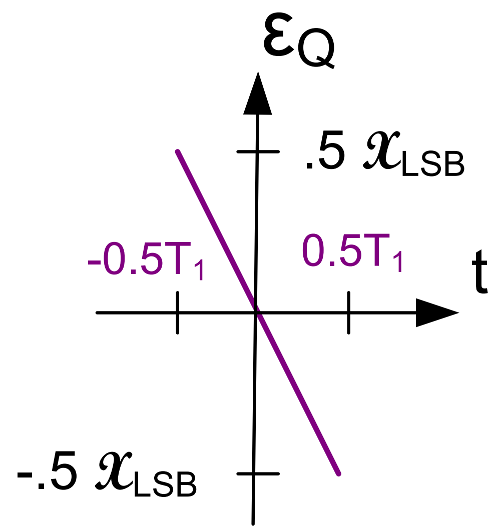
Quantization Noise in ADC

$$E_{RMS} = \sqrt{\frac{1}{T_1}\int_{-T_1/2}^{T_1/2} \varepsilon_Q{}^2(t)\,dt}$$

$$E_{RMS} = \sqrt{\frac{1}{T_1}\int_{-T_1/2}^{T_1/2} \left(-\frac{x_{LSB}}{T_1}\right)^2 t^2\,dt}$$

$$E_{RMS} = x_{LSB}\sqrt{\frac{1}{T_1^3}\left.\frac{t^3}{3}\right|_{-T_1/2}^{T_1/2}}$$

$$E_{RMS} = \frac{x_{LSB}}{\sqrt{12}}$$



$$\varepsilon_Q(t) = -\left(\frac{X_{LSB}}{T_1}\right)t$$

Quantization Noise in ADC

$$E_{RMS} = \frac{x_{LSB}}{\sqrt{12}}$$

The signal to quantization noise ratio (SNR) can now be determined. Since the input signal is a sawtooth waveform of period T and amplitude X_{REF}, it follows by the same analysis that it has an RMS value of

$$x_{RMS} = \frac{x_{REF}}{\sqrt{12}}$$

Thus the SNR is given by

$$SNR = \frac{x_{RMS}}{E_{RMS}} = \frac{x_{RMS}}{x_{LSB}} = 2^n$$

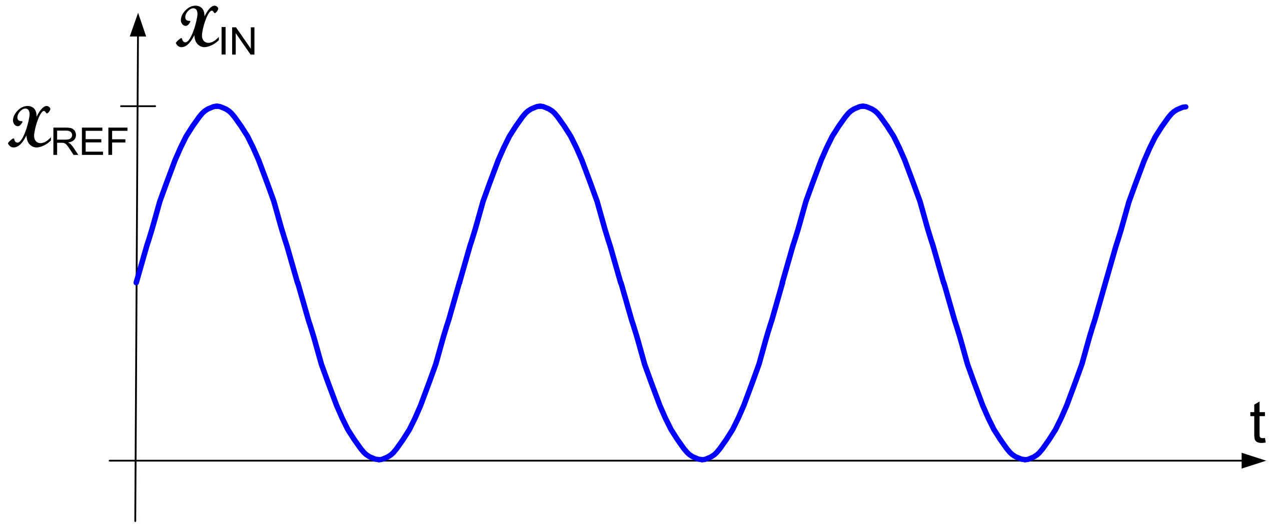
or, in dB,

$$SNR_{dB} = 20\left(n \bullet \log 2\right) = 6.02n$$

Note: dB subscript often neglected when not concerned about confusion

Quantization Noise in ADC

How does the SNR change if the input is a sinusoid that goes from 0 to x_{REF} centered at $x_{REF}/2$?



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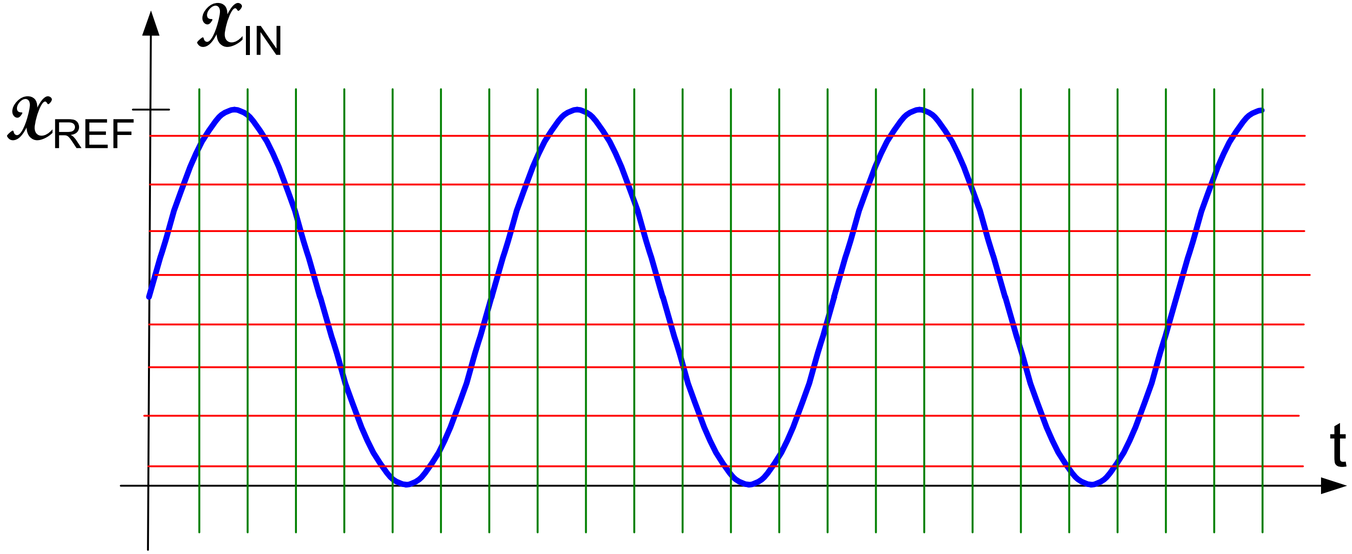
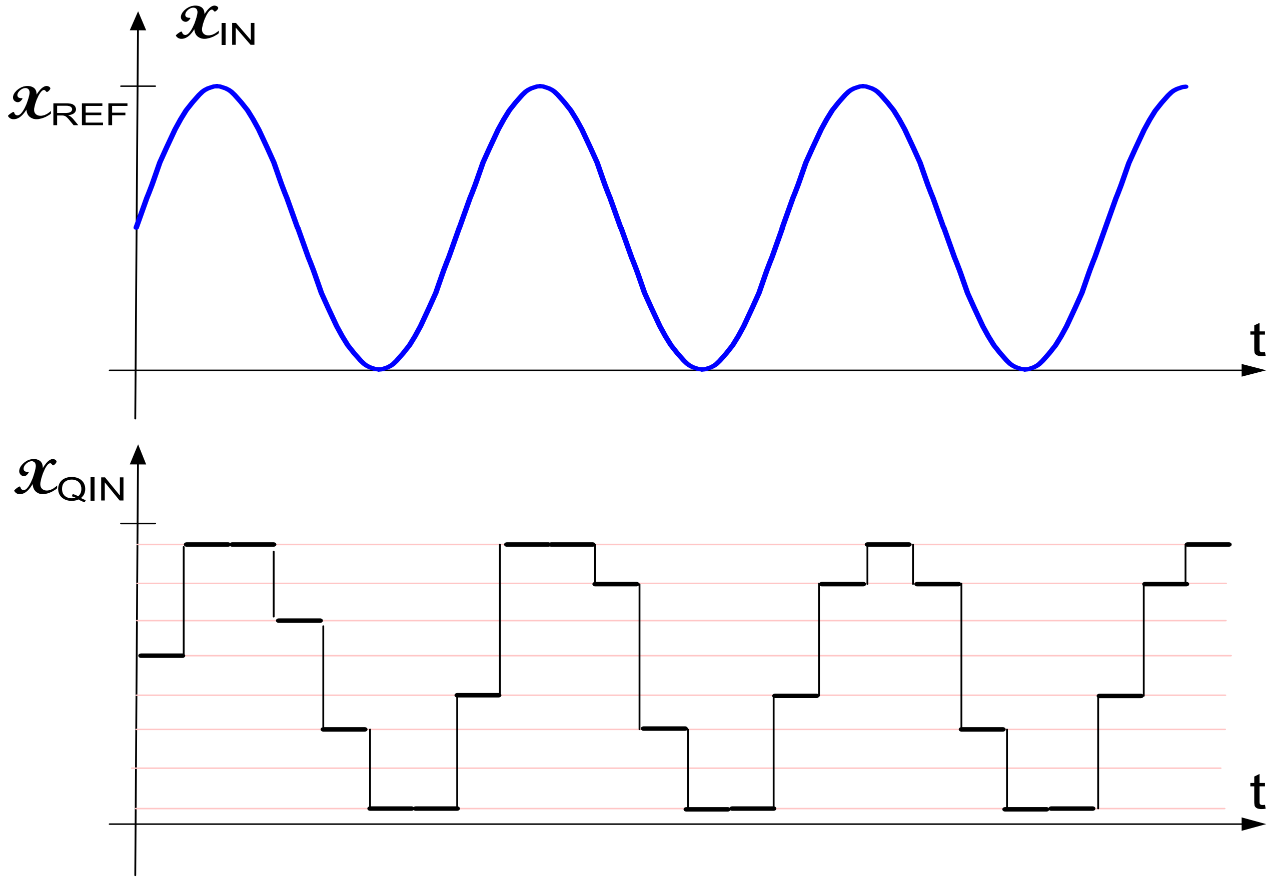


Time and amplitude quantization points

Quantization Noise in ADC

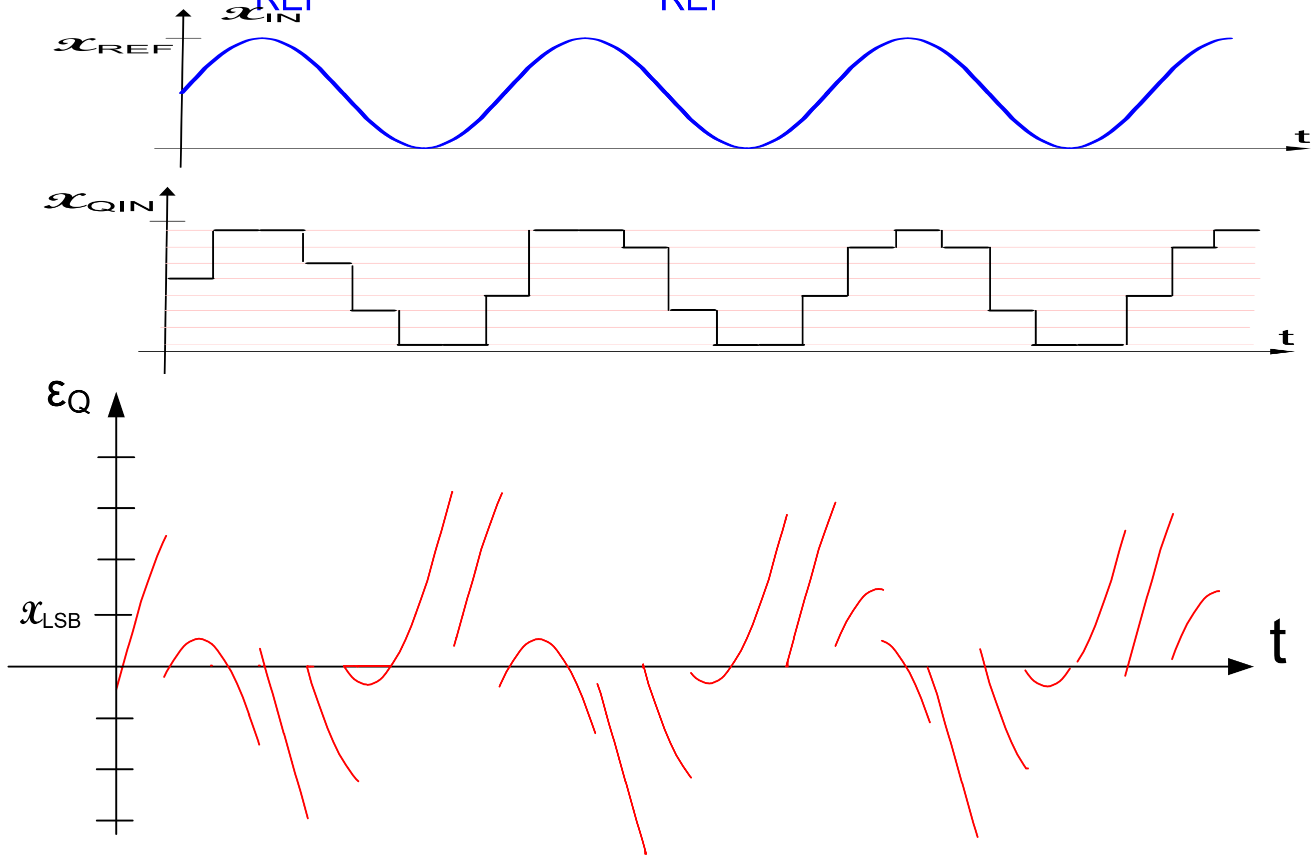
How does the SNR change if the input is a sinusoid that goes from 0 to $\mathcal{X}_{REF}$ centered at $\mathcal{X}_{REF}/2$?



Time and Amplitude Quantized Waveform

Quantization Noise in ADC

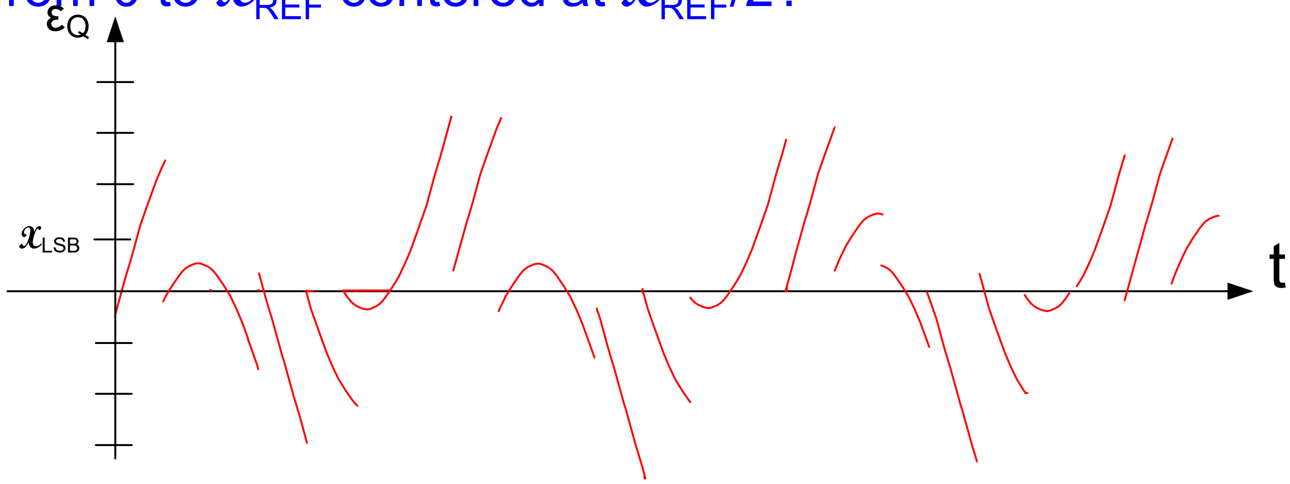
How does the SNR change if the input is a sinusoid that goes from 0 to $\mathcal{X}_{REF}$ centered at $\mathcal{X}_{REF}/2$?



Error waveform

Quantization Noise in ADC

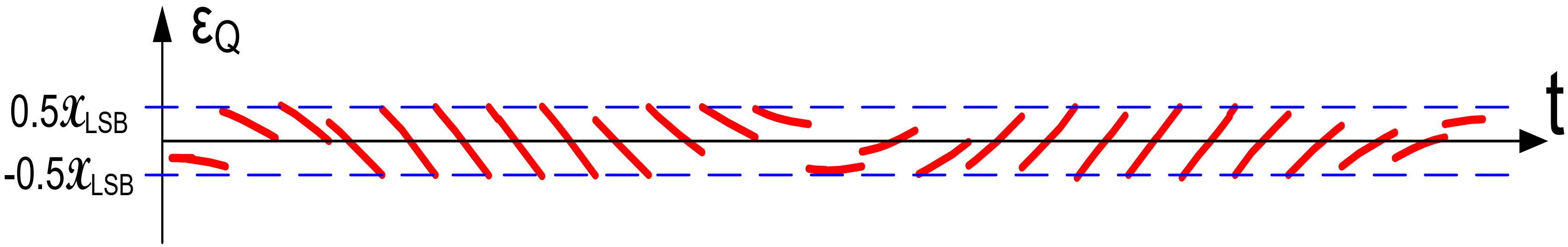
How does the SNR change if the input is a sinusoid that goes from 0 to $\mathcal{X}_{REF}$ centered at $\mathcal{X}_{REF}/2$?



- Appears to be highly uncorrelated with input even though deterministic
- Mathematical expression for ε_Q very messy
- Excursions exceed X_{LSB} (but will be smaller and bounded by $\pm X_{LSB}/2$ for lower frequency signal/frequency clock ratios)
- For lower frequency inputs and higher resolution, at any time, errors are approximately uniformly distributed between $-X_{LSB}/2$ and $X_{LSB}/2$
- Analytical form for ε_{QRMS} essentially impossible to obtain from $\varepsilon_Q(t)$

Quantization Noise in ADC

How does the SNR change if the input is a sinusoid that goes from 0 to $\mathcal{X}_{REF}$ centered at $\mathcal{X}_{REF}/2$?



For low f_{SIG}/f_{CL} ratios, bounded by ±XLB and at any point in time, behaves almost as if a uniformly distributed random variable

$$\varepsilon_Q \sim U[-0.5X_{LSB}, 0.5X_{LSB}]$$

Quantization Noise in ADC

Recall:

If the random variable f is uniformly distributed in the interval [A,B]
$f : U[A,B]$ then the mean and standard deviation of f are given by

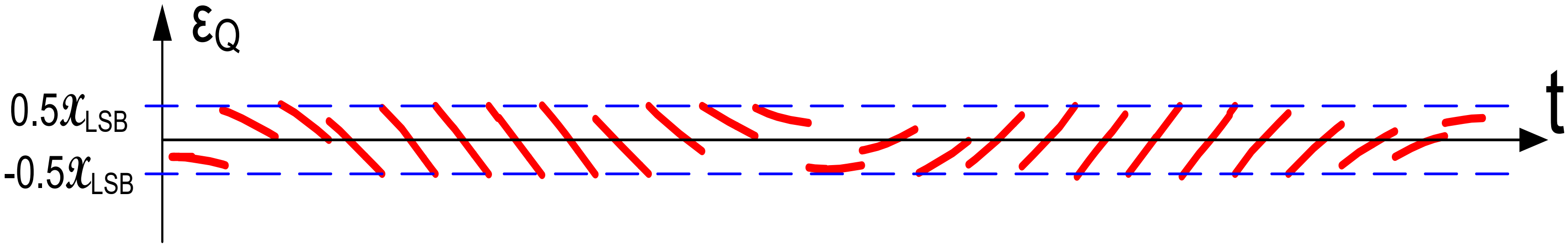
$$\mu_f = \frac{A+B}{2} \qquad \sigma_f = \frac{B-A}{\sqrt{12}}$$

Theorem: If n(t) is a random process, then for large T,

$$V_{RMS} = \sqrt{\frac{1}{T}\int_{t_1}^{t_1+T} n^2(t)\,dt} = \sqrt{\sigma_n^2 + \mu_n^2}$$

Quantization Noise in ADC

How does the SNR change if the input is a sinusoid that goes from 0 to $\mathcal{X}_{REF}$ centered at $\mathcal{X}_{REF}/2$?



$$\varepsilon_Q \sim U[-0.5X_{LSB},\ 0.5X_{LSB}]$$

$$\mu_{\varepsilon_Q} = \frac{A+B}{2} = 0 \qquad \sigma_f = \frac{B-A}{\sqrt{12}} = \frac{X_{LSB}}{\sqrt{12}}$$

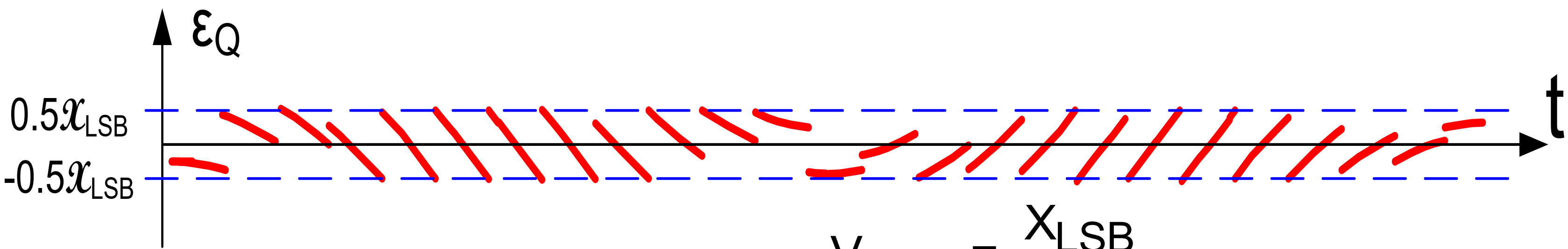
$$V_{RMS} = \sqrt{\frac{1}{T}\int_{t_1}^{t_1+T} n^2(t)\,dt} = \sqrt{\sigma_n^2 + \mu_n^2}$$

$$V_{RMS} = \sigma_{\varepsilon_Q} = \frac{X_{LSB}}{\sqrt{12}}$$

Note this is the same RMS noise that was present with a triangular input

Quantization Noise in ADC

How does the SNR change if the input is a sinusoid that goes from 0 to $\mathcal{X}_{REF}$ centered at $\mathcal{X}_{REF}/2$?



$$V_{RMS} = \frac{X_{LSB}}{\sqrt{12}}$$

But $$V_{INRMS} = \left(\frac{X_{REF}}{2}\right)\frac{1}{\sqrt{2}}$$

Thus obtain

$$SNR = \frac{\dfrac{X_{REF}}{2\sqrt{2}}}{\dfrac{X_{LSB}}{\sqrt{12}}} = 2^n\sqrt{\frac{3}{2}}$$

Finally, in db,

$$SNR_{dB} = 20\log\left(2^n\sqrt{\frac{3}{2}}\right) = 6.02\,n + 1.76$$

ENOB based upon Quantization Noise

$$SNR = 6.02\,n + 1.76$$

Solving for n, obtain

$$ENOB = \frac{SNR_{dB} - 1.76}{6.02}$$

Note: could have used the SNR_{dB} for a triangle input and would have obtained the expression

$$ENOB = \frac{SNR_{dB}}{6.02}$$

But the earlier expression is more widely used when specifying the ENOB based upon the noise level present in a data converter

ENOB based upon Quantization Noise

For very low resolution levels, the assumption that the quantization noise is uncorrelated with the signal is not valid and the ENOB expression will cause a modest error

from van de Plassche (p13)

$$SNR_{corr} \cong \left(2^n - 2 + \frac{4}{\pi}\right)\sqrt{\frac{3}{2}}$$

Res (n)	SNR_{corr}	SNR
1	3.86	7.78
2	12.06	13.8
3	19.0	19.82
4	25.44	25.84
5	31.66	31.86
6	37.79	37.88
8	49.90	49.92
10	61.95	61.96

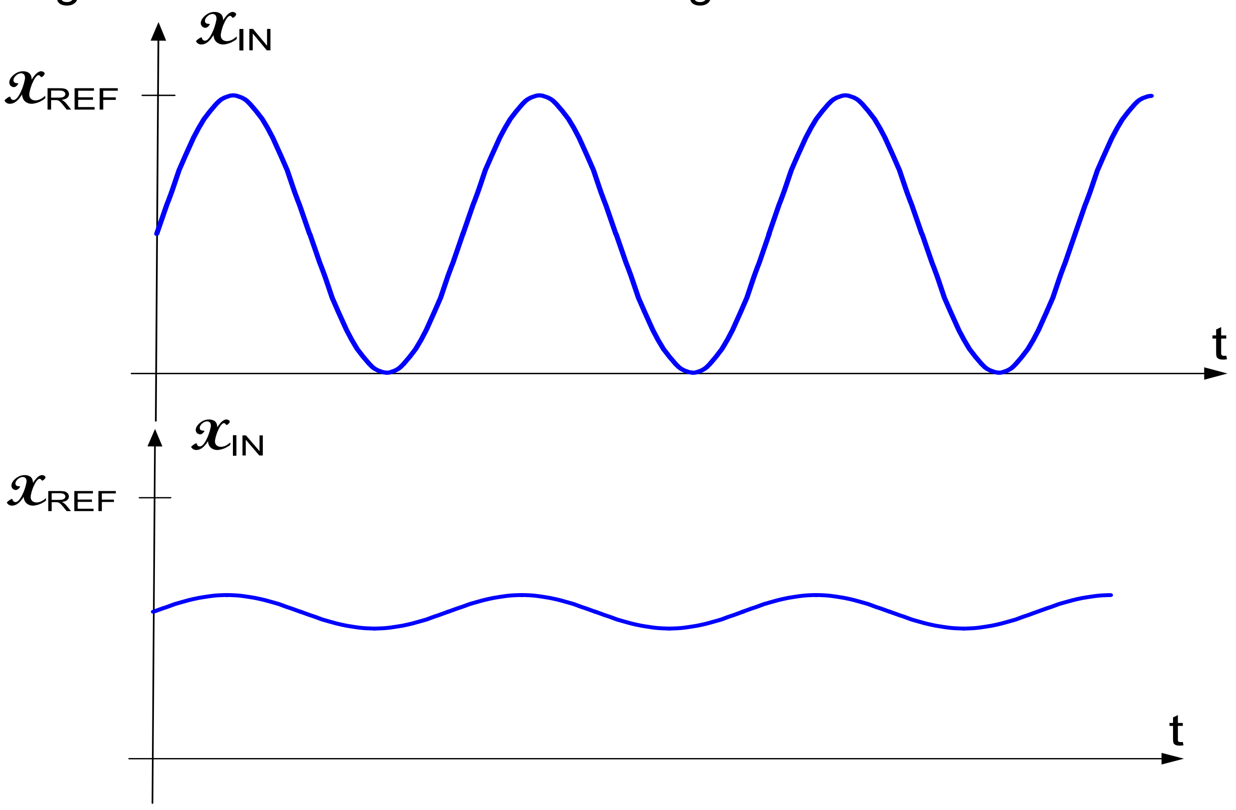
SNR = 6.02 n +1.76

Table values in dB

Almost no difference for n ≥ 3

Quantization Noise

Effects of quantization noise can be very significant, even at high resolution, when signals are not of maximum magnitude

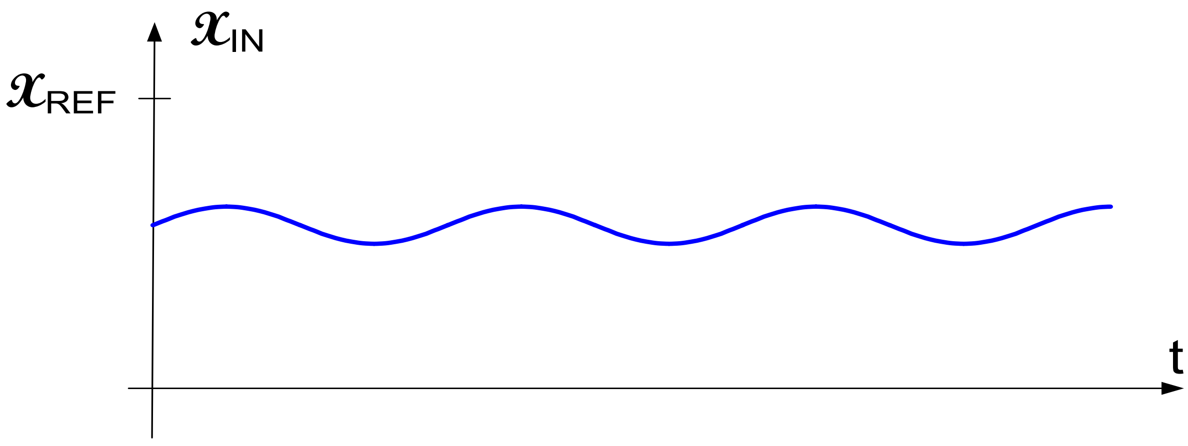


Quantization noise remains constant but signal level is reduced

The desire to use a data converter at a small fraction of full range
is one of the major reasons high resolution is required in many applications

Quantization Noise

Effects of quantization noise can be very significant, even at high resolution, when signals are not of maximum magnitude



Quantization Noise

Example: If a 14-bit audio output is derived from a DAC designed for providing an output of 100W but the normal listening level is at 50mW, what is the SNR due to quantization noise at maximum output and at the normal listening level? What is the ENOB of the audio system when operating at 50mW?

At 100W output, SNR=6.02n+1.76 = 90.6dB

$$\frac{V^2}{R_L}=100W \qquad \frac{V_1^2}{R_L}=50mW \qquad V_1=\frac{V}{44.7}$$

$$20\log_{10}V_1=20\log_{10}V-20\log_{10}44.7=20\log_{10}V\ \ -33dB$$

At 50mW output, SNR reduced by 33dB to 57.6dB

$$ENOB = \frac{SNR_{dB}-1.76}{6.02} = \frac{57.6-1.76}{6.02} = -9.3$$

Note the dramatic reduction in the effective resolution of the DAC when operated at only a small fraction of full-scale.

ENOB Summary

Resolution:

$$\text{ENOB} = \frac{\log_{10} N_{ACT}}{\log_{10} 2} = \log_2 N_{ACT}$$

INL:

$$\text{ENOB} = n_R - \log_2(\upsilon) - 1$$

n_R specified res, v INL in LSB

$$\text{ENOB} = -\log_2(\text{INL}_{REF}) - 1$$

INL_{REF} INL rel to X_{REF}

DNL:

HW problem

Quantization noise:

$$\text{ENOB} = \frac{\text{SNR}_{dB}}{6.02}$$

rel to triangle/sawtooth

$$\boxed{\text{ENOB} = \frac{\text{SNR}_{dB} - 1.76}{6.02}}$$

rel to sinusoid

Most widely used for static characteristics

Additional ENOB will be introduced when discussing dynamic characteristics

Performance Characterization of Data Converters

- Static characteristics
  - Resolution
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  - Absolute Accuracy
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  - Low-f Spurious Free Dynamic Range (SFDR)
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  - Power Dissipation

Absolute Accuracy

Absolute Accuracy is the difference between the actual output and the ideal or desired output of a data converter

The ideal or desired output is in reference to an absolute standard (often maintained by the National Institute of Standards and Technology – NIST) (renamed from National Bureau of Standards in 1988) and could be volts, amps, time, weight, distance, or one of a large number of other physical quantities)

Absolute accuracy provides no tolerance to offset errors, gain errors, nonlinearity errors, quantization errors, frequency rolloff, or noise

In many applications, absolute accuracy is not of a major concern

Absolute accuracy generally dominated by the nonidealities of the reference (a data converter is a ratio-metric device so no fundamental limit on ratio portion)

but … scales, meters, etc. may be more concerned about absolute accuracy than any other parameter

Relative Accuracy

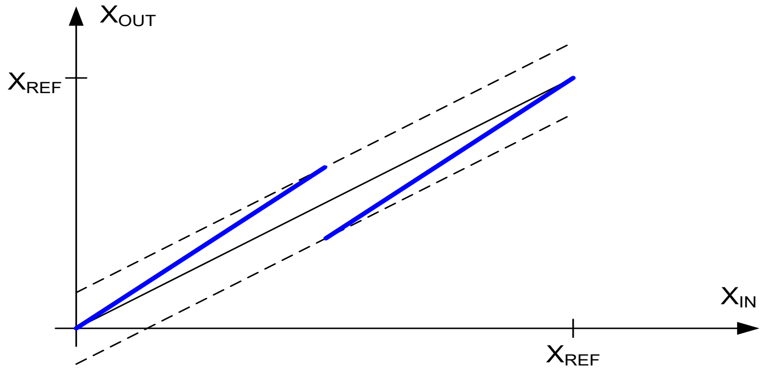
In the context of data converters, pseudo-static Relative Accuracy is the difference between the actual output and an appropriate fit-line to overall output of the data converter

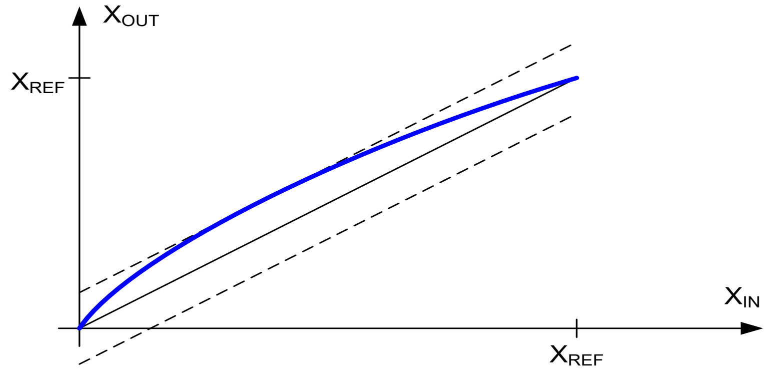
INL is often used as a measure of the relative accuracy

In many, if not most, applications, relative accuracy is of much more concern than absolute accuracy

Some architectures with good relative accuracy will have very small deviations in the outputs for closely-spaced inputs whereas others may have relatively large deviations in outputs for closely-spaced inputs

DNL provides some measure of how outputs for closely-spaced inputs compare





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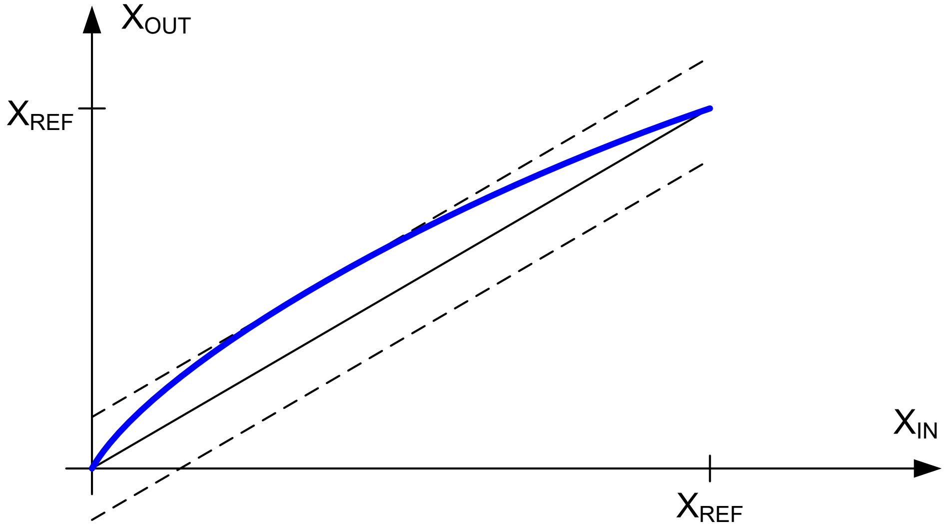
Limitations of INL & DNL in Characterizing Linearity

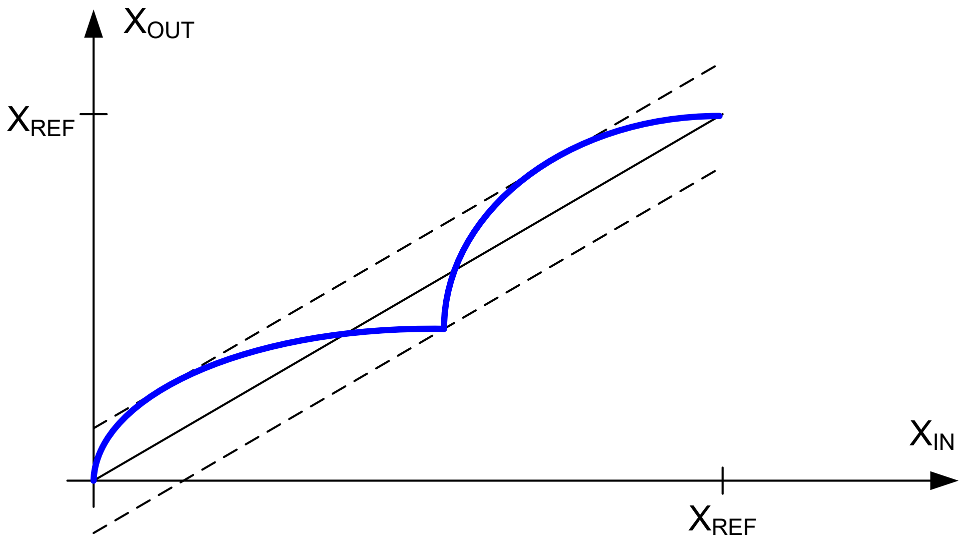
- **INL is a key parameter that is attempting to characterize the overall linearity of a DAC !**
- **INL is a key parameter that is attempting to characterize the overall linearity of an ADC !**
- **DNL is a key parameter that is attempts to characterize the local linearity of a DAC !**
- **DNL is a key parameter that is attempts to characterize the local linearity of an ADC !**

Are INL and DNL effective at characterizing the linearity of a data converter?

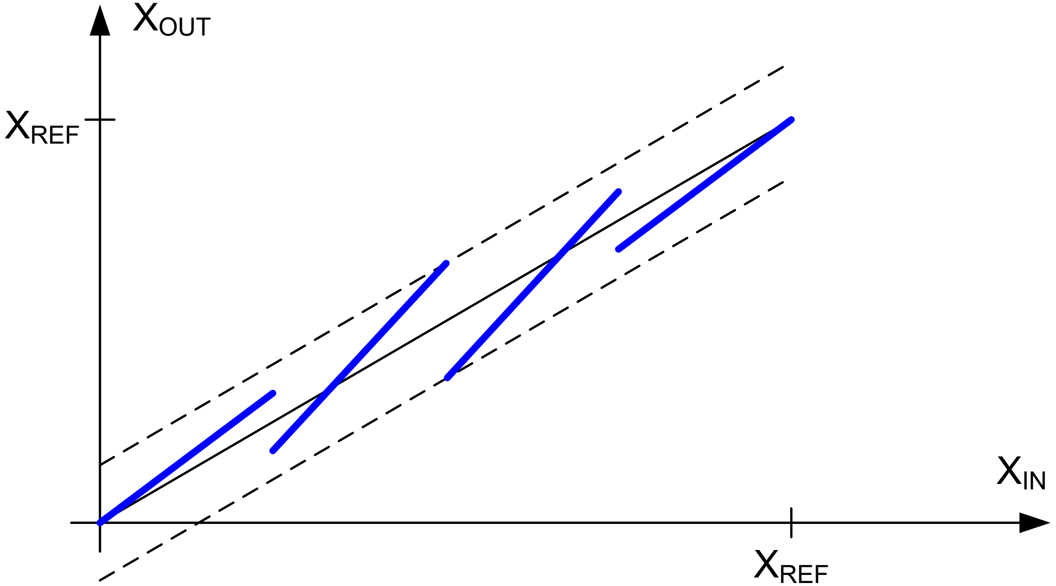
Limitations of INL & DNL in Characterizing Linearity

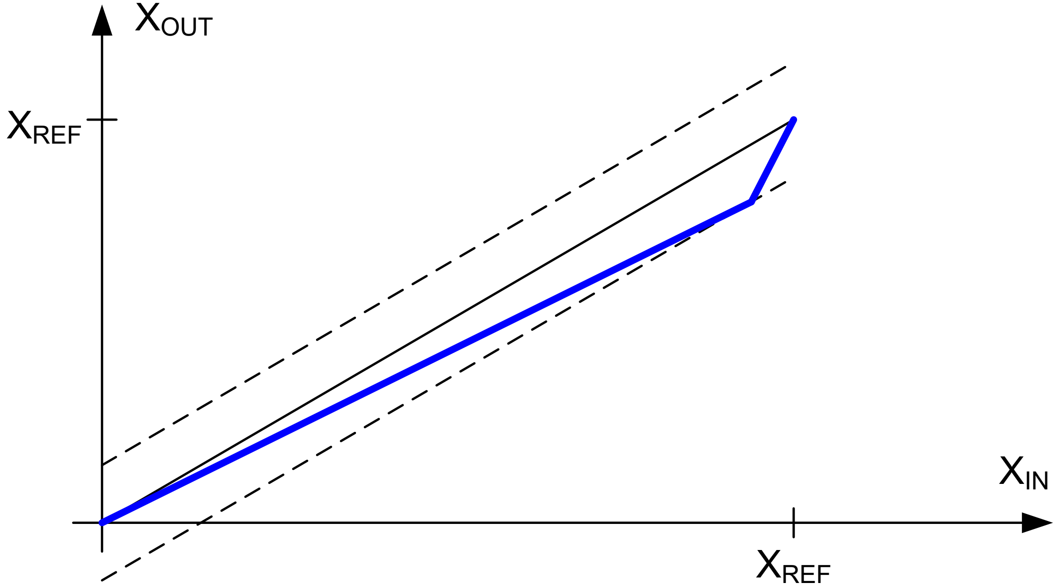
Consider the following 4 transfer characteristics, all of which have the same INL



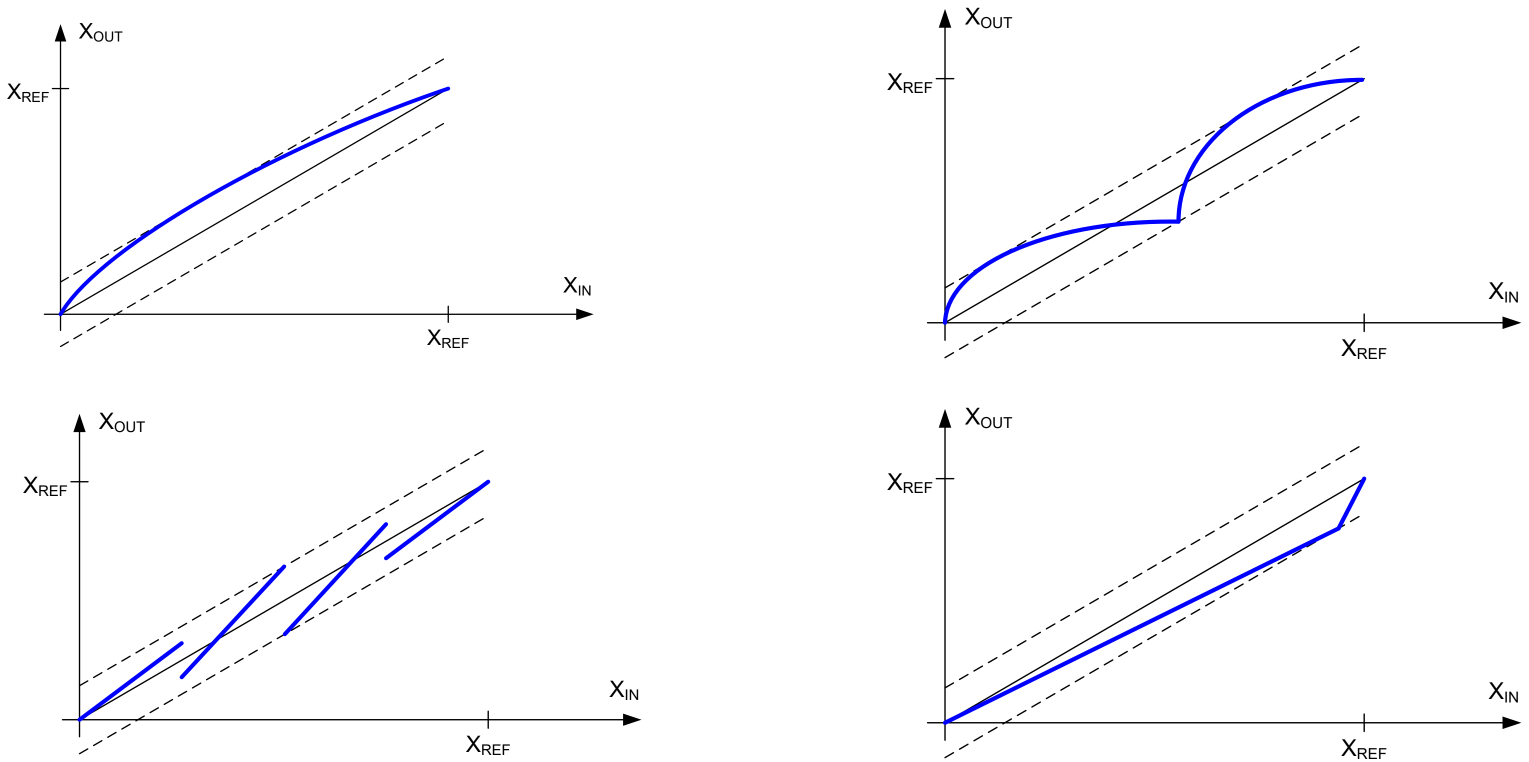


Limitations of INL & DNL in Characterizing Linearity





Limitations of INL & DNL in Characterizing Linearity



Although same INL, dramatic difference in performance particularly when inputs are sinusoidal-type excitations

INL also gives little indication of how performance degrades at higher frequencies

Spectral Analysis often used as an alternative (and often more useful in many applications) linearity measure for data converters

Linearity Issues

- INL is often not adequate for predicting the linearity performance of a data converter
- Distortion (or lack thereof) is of major concern in many applications
- Distortion is generally characterized in terms of the harmonics that may appear in a waveform

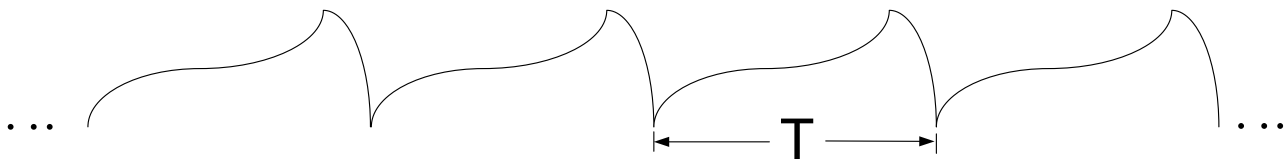
Spectral Analysis often used as an alternative (and often more useful in many applications) linearity measure for data converters

Performance Characterization of Data Converters

- Static characteristics
  - Resolution
  - Least Significant Bit (LSB)
  - Offset and Gain Errors
  - Absolute Accuracy
  - Relative Accuracy
  - Integral Nonlinearity (INL)
  - Differential Nonlinearity (DNL)
  - Monotonicity (DAC)
  - Missing Codes (ADC)
  - Quantization Noise
  - Low-f Spurious Free Dynamic Range (SFDR)
  - Low-f Total Harmonic Distortion (THD)
  - Effective Number of Bits (ENOB)
  - Power Dissipation

Spectral Characterization

Spectral Analysis



If f(t) is periodic

$$f(t) = A_0 + \sum_{k=1}^{\infty} A_k \sin(k\omega t + \theta_k)$$

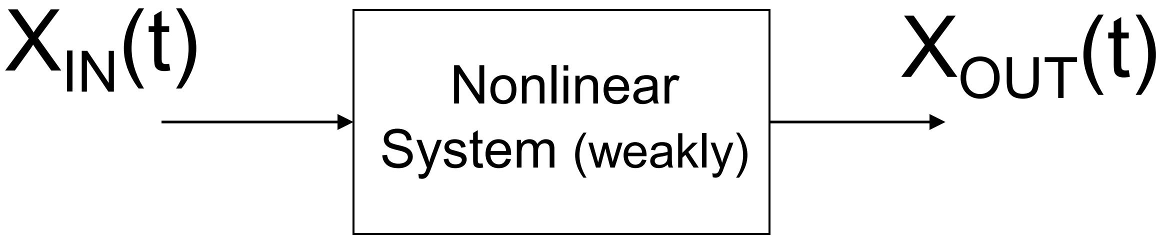
alternately

$$f(t) = A_0 + \sum_{k=1}^{\infty} a_k \sin(k\omega t) + \sum_{k=1}^{\infty} b_k \cos(k\omega t) \qquad \omega = \frac{2\pi}{T}$$

$$A_k = \sqrt{a_k^2 + b_k^2}$$

Termed the Fourier Series Representation of f(t)

Spectral Analysis



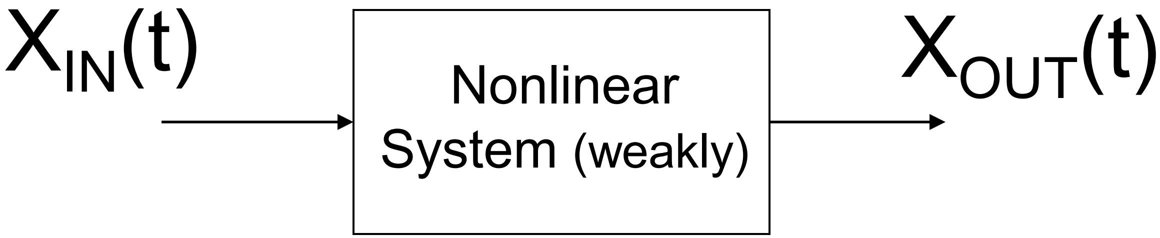
Often the system of interest is ideally linear but practically it is weakly nonlinear.

Often the input is nearly periodic and often sinusoidal and in latter case desired output is also sinusoidal

Weak nonlinearity will cause harmonic distortion (often just termed distortion) of signal as it is propagated through the system

Spectral analysis often used to characterize effects of the weak nonlinearity

Spectral Analysis



Distortion Types:

Frequency Distortion

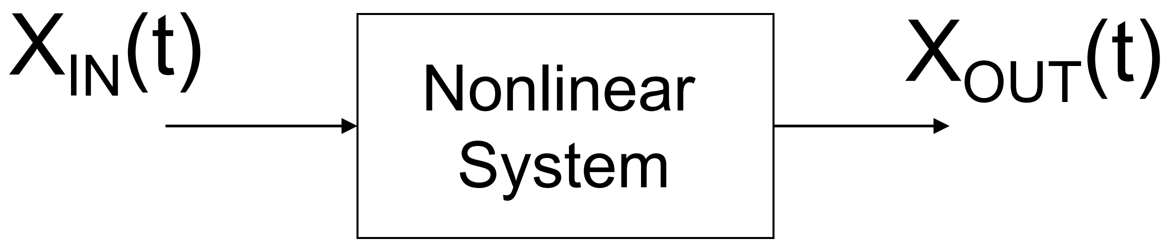
Nonlinear Distortion (alt. harmonic distortion)

Frequency Distortion: Amplitude and phase of system is altered but output is linearly related to input

Nonlinear Distortion: System is not linear, frequency components usually appear in the output that are not present in the input

Spectral Analysis is the characterization of a system with a periodic input with the Fourier series relationships between the input and output waveforms

Spectral Analysis



If $$X_{IN}(t) = X_m \sin(\omega t + \theta)$$

$$X_{OUT}(t) = A_0 + \sum_{k=1}^{\infty} A_k \sin(k\,\omega\,t + \theta_k)$$

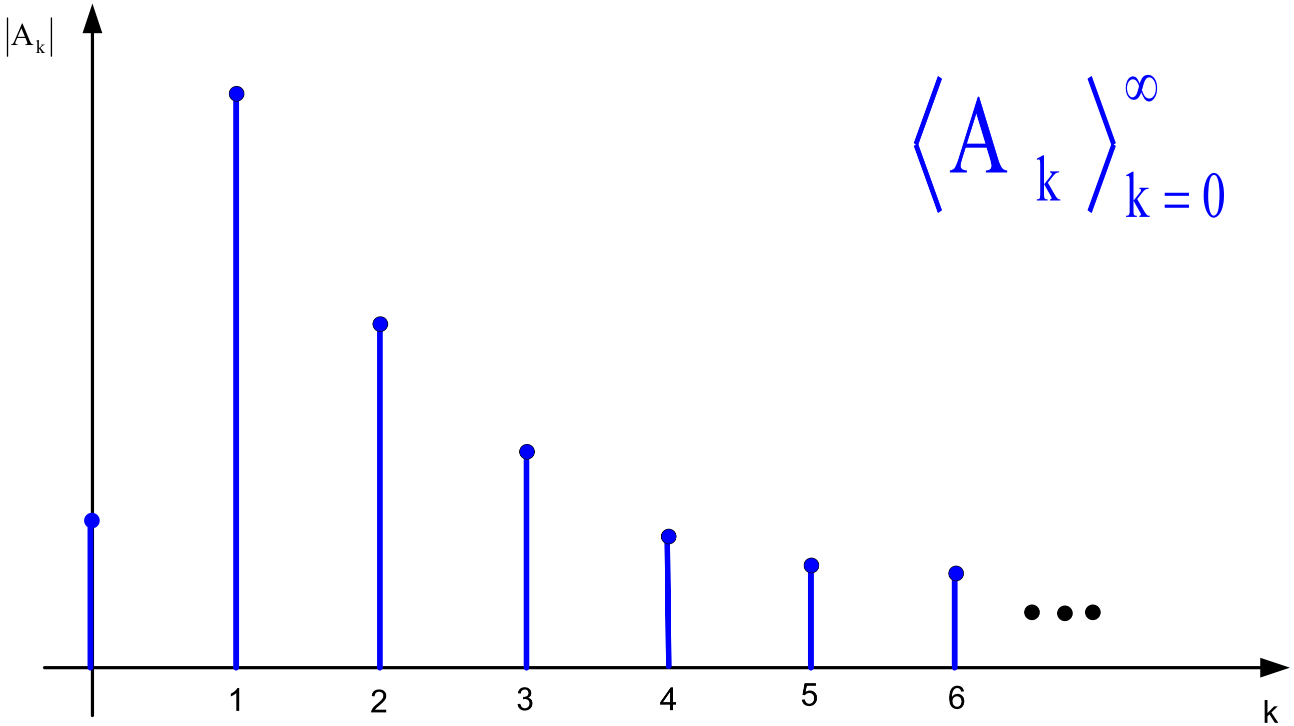
All spectral performance metrics depend upon the sequences $\left\langle A_k \right\rangle_{k=0}^{\infty}$ $\left\langle \theta_k \right\rangle_{k=1}^{\infty}$
(index sequence, not time sequence)

Typical spectral performance metrics of interest: SNDR, SDR, THD, SFDR, IMOD

Alternately

$$X_{OUT}(t) = A_0 + \sum_{k=1}^{\infty} a_k \sin(k\omega t) + \sum_{k=1}^{\infty} b_k \cos(k\omega t) \qquad A_k = \sqrt{a_k^2 + b_k^2} \qquad \theta_k = \tan^{-1}\left(\frac{b_k}{a_k}\right)$$

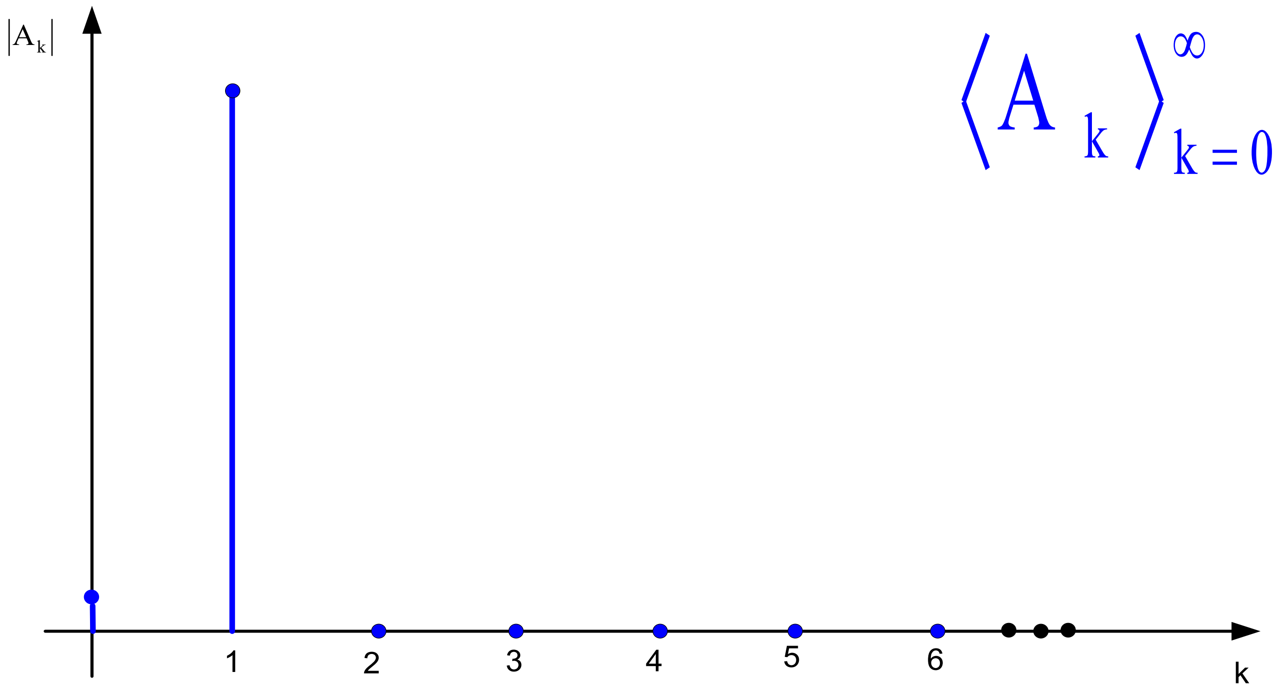
Distortion Analysis



- Often termed the DFT coefficients (will show later)
- Spectral lines, not a continuous function

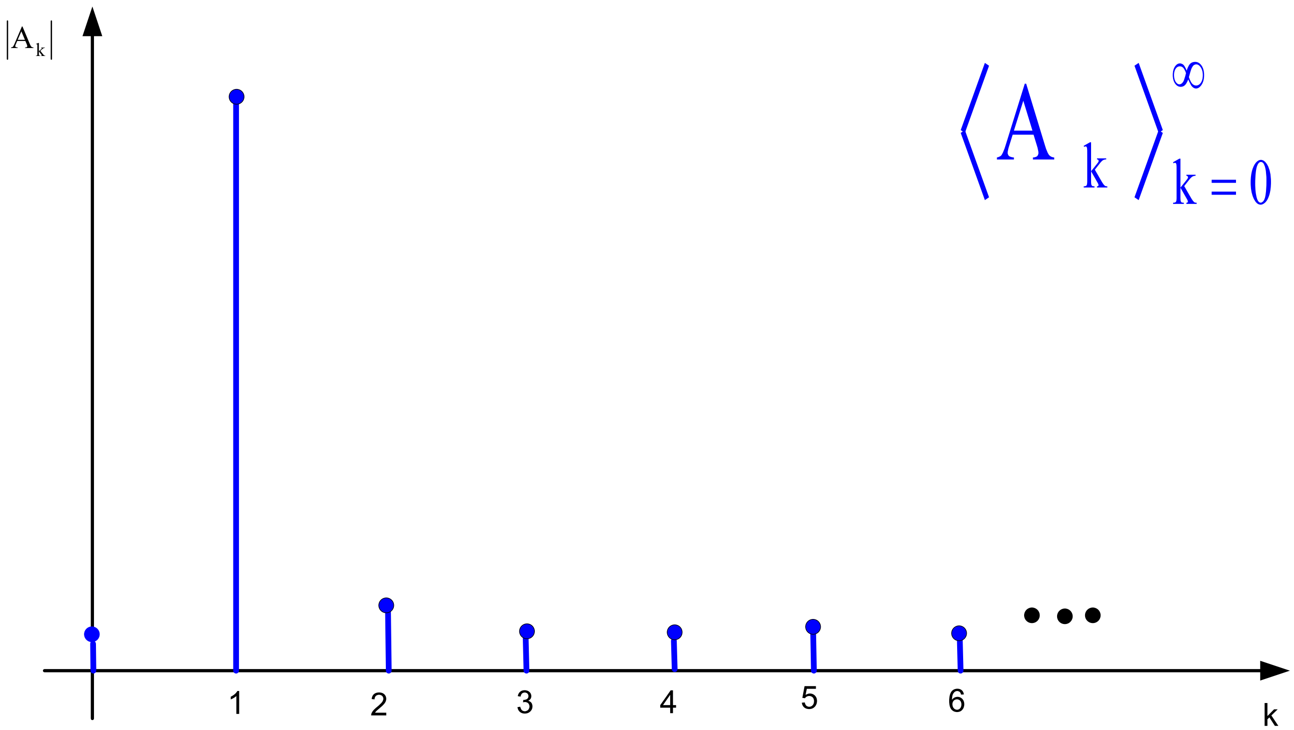
A_1 is termed the fundamental (when input is sinusoid or periodic)
A_k is termed the kth harmonic (when input is sinusoid or periodic)

Distortion Analysis



Often ideal response will have only fundamental present and all remaining spectral terms will vanish

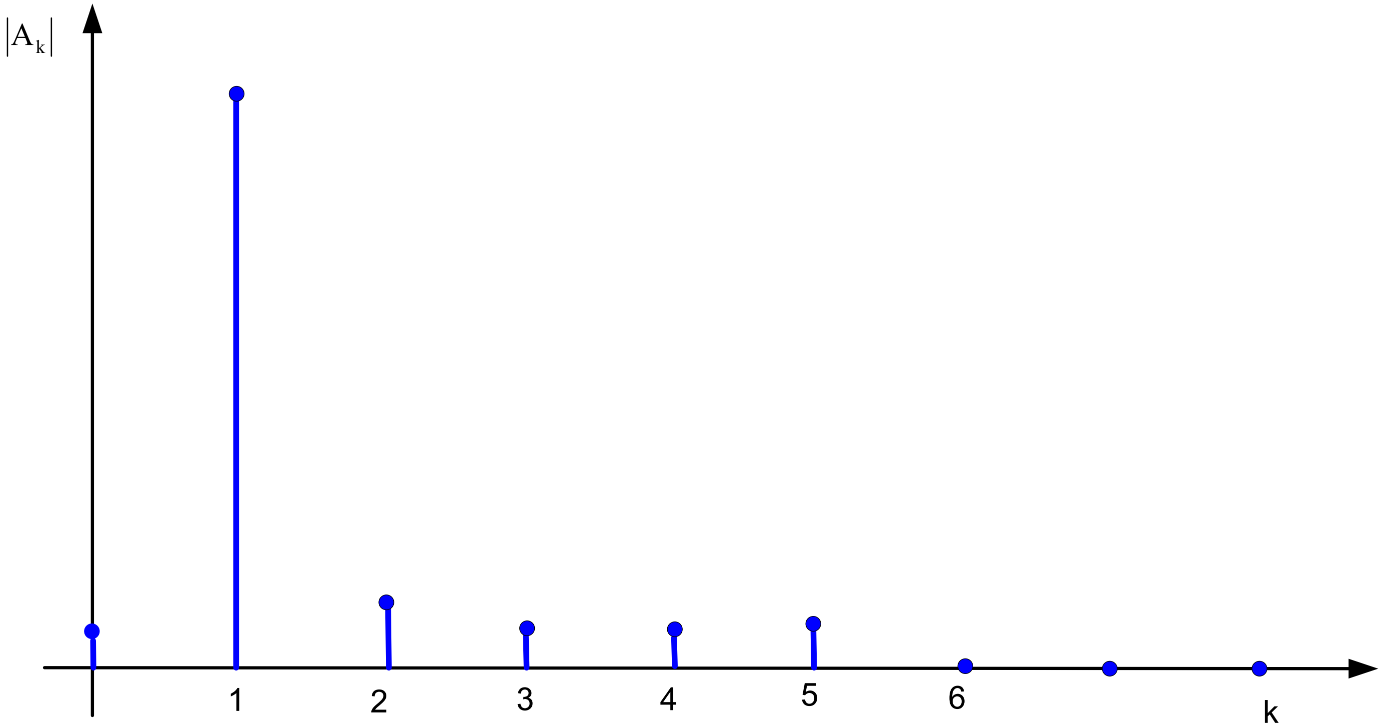
Distortion Analysis



For a low distortion signal, the 2nd and higher harmonics are generally much smaller than the fundamental

The magnitude of the harmonics generally decrease rapidly with k for low distortion signals

Distortion Analysis



Assume f(t) is periodic with period $T = \frac{1}{f}$

f(t) is band-limited to frequency $2\pi f k_X$ if A_k=0 for all $k>k_x$

where $\langle A_k \rangle_{k=0}^{\infty}$ are the Fourier series coefficients of f(t)

Distortion Analysis

Total Harmonic Distortion, THD

$$\mathrm{THD} = \frac{\text{RMS voltage in harmonics}}{\text{RMS voltage of fundamental}}$$

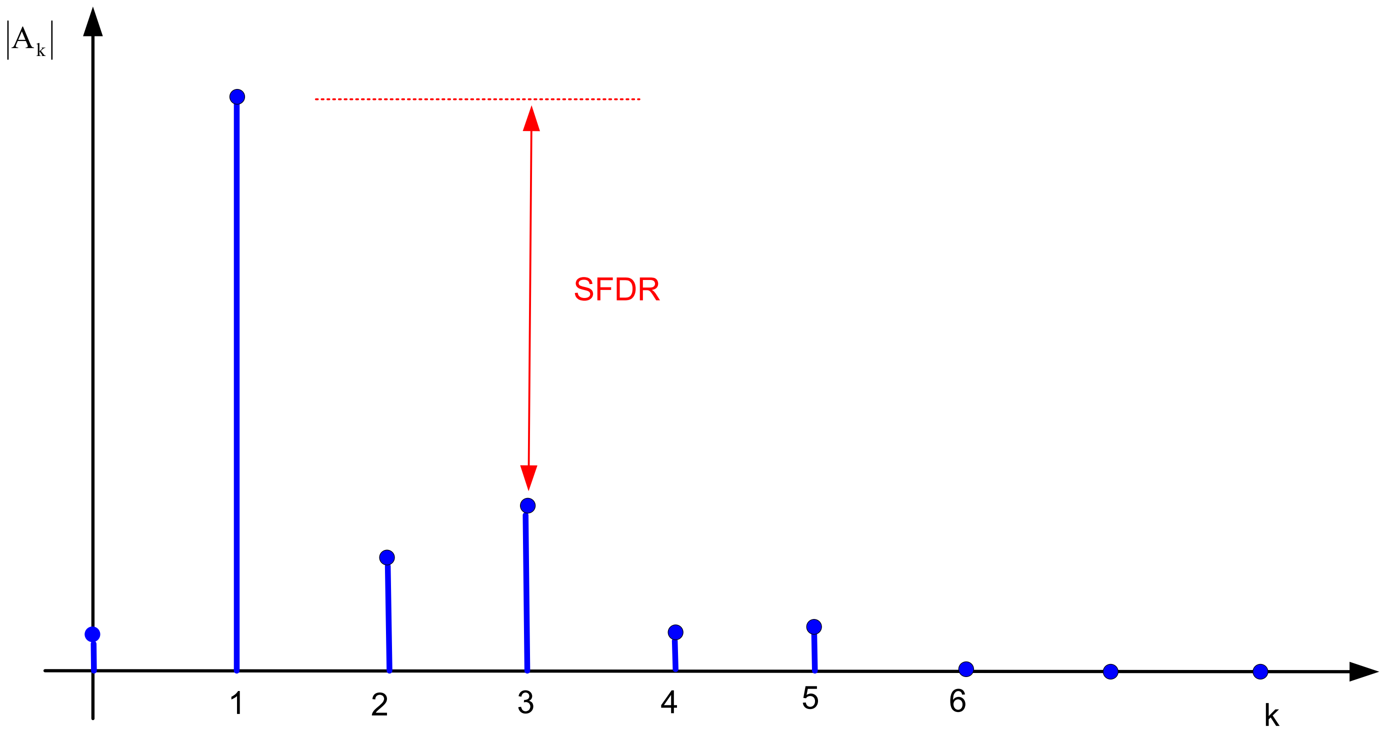
$$\mathrm{THD} = \frac{\sqrt{\left(\frac{A_2}{\sqrt{2}}\right)^2 + \left(\frac{A_3}{\sqrt{2}}\right)^2 + \left(\frac{A_4}{\sqrt{2}}\right)^2 + \ldots}}{\frac{A_1}{\sqrt{2}}}$$

$$\mathrm{THD} = \frac{\sqrt{\sum_{k=2}^{\infty} A_k^2}}{A_1}$$

Distortion Analysis

Spurious Free Dynamic Range, SFDR

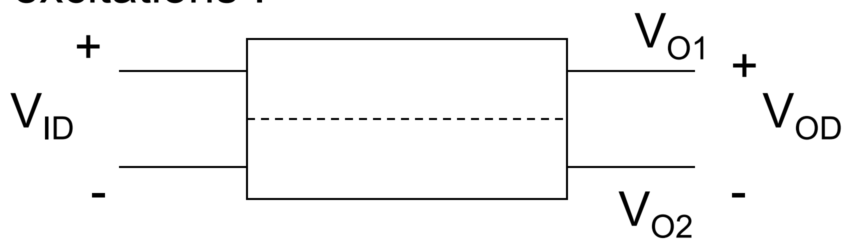
The SFDR is the difference between the fundamental and the largest harmonic



SFDR and THD are usually determined by either the second or third harmonic

Distortion Analysis

Theorem: In a fully differential symmetric circuit, all even-order terms are absent in the Taylor's series output for symmetric differential sinusoidal excitations !



Proof: Expanding in a Taylor's series around V_{ID}=0, we obtain

$$V_{01} = f\left(V_{ID}\right) = \sum_{k=0}^{\infty} h_k V_{ID}^k$$

$$V_{02} = f\left(-V_{ID}\right) = \sum_{k=0}^{\infty} h_k \left(-V_{ID}\right)^k$$

$$V_{OD} = V_{01} - V_{02} = \sum_{k=0}^{\infty} h_k \left(V_{ID}\right)^k - \sum_{k=0}^{\infty} h_k \left(-V_{ID}\right)^k$$

$$V_{OD} = \sum_{k=0}^{\infty} h_k \left[\left(V_{ID}\right)^k - \left(-V_{ID}\right)^k\right]$$

$$V_{OD} = \sum_{k=0}^{\infty} h_k \left[\left(V_{ID}\right)^k - \left(-1\right)^k \left(V_{ID}\right)^k\right]$$

When k is even, the corresponding term in [ ] vanishes

Stay Safe and Stay Healthy !

End of Lecture 4